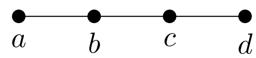
Chapter 1 Graphs and Isomorphisms

A **simple graph** G is an ordered pair (V, E) of **vertices** $V = V(G)$ and **edges** $E = E(G)$, where $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$. $\{u, v\}$ can be simplified as uv.

$|V|$ is **order** of G $|E|$ is **size** of G

1: What are the order and the size of $G = (\{a, b, c, d\}, \{ab, bc, cd\})$.



Solution: order is 4, size is 3

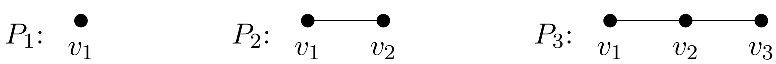
Definition For $u, v \in V$ and $e \in E$, we define

- If $uv \in E$, then u and v are **adjacent** and called **neighbors**.
- If $v \in e$, then v and e are **incident**.
- Edges are **adjacent** if they have a vertex in common.

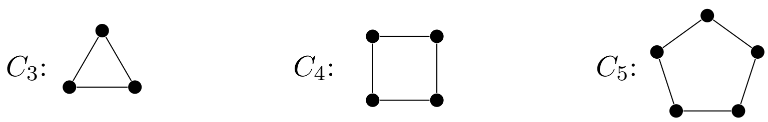
Drawing of G *assigns point* to V and *curves* to E, where the endpoints of uv are u and v.

If $V(G) = \emptyset$ then G is a *null* graph, sometimes called *empty graph*.

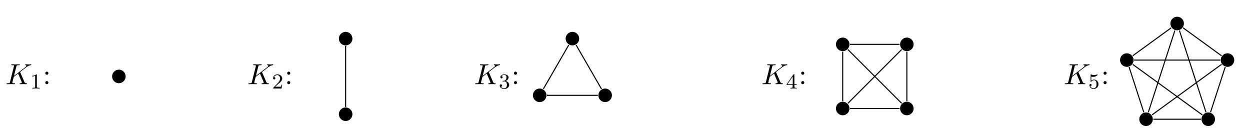
A **path** P_n of length $n - 1$ has distinct vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for all $1 \leq i \leq n - 1$.



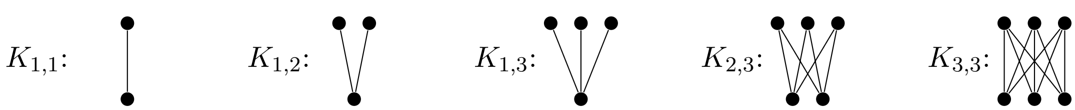
A **cycle** C_n of length n is obtained from $P_n = v_1, \ldots, v_n$ by adding edge $v_1 v_n$



A **complete** graph K_n has n vertices and for all $u, v \in V(K_n), uv \in E(K_n)$, i.e. *all* edges.



A **complete bipartite** graph $K_{n,m}$ has $V = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_m\}$ and $E = \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq m\}$.



2: What is $|E(P_n)|$, $|E(C_n)|$, $|E(K_n)|$, $|E(K_{n,m})|$?

Solution: $|E(P_n)| = n - 1$, $|E(C_n)| = n$, $|E(K_n)| = \binom{n}{2}$, $|E(K_{n,m})| = nm$

Graphs G and H are **isomorphic**, denoted by $G \cong H$ if there exists a bijective mapping $\varphi : V(G) \to V(H)$ such that

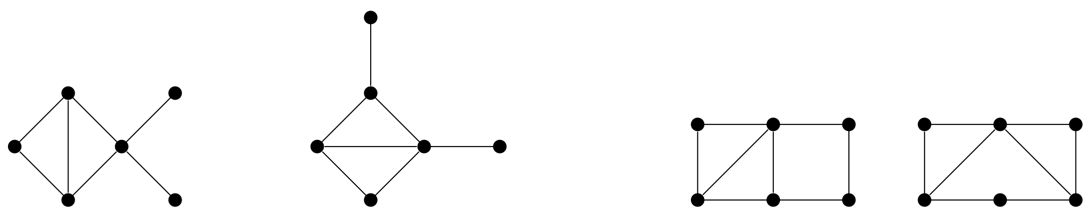
$$uv \in E(G) \text{ if and only if } \varphi(u)\varphi(v) \in E(H) \text{ for all } u, v \in V(G).$$

3: Find an isomorphims between the following two graphs

Solution: $\varphi(u_0) = a$, $\varphi(u_1) = c$, $\varphi(u_2) = \gamma$, $\varphi(u_3) = b$, $\varphi(u_4) = f$

Graphs G and H are **non-isomorphic** if they are not isomorphic.

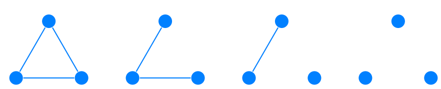
4: Decide if the following pairs are isomorphic and justify your answer (i.e. find an isomorphism or say why they are non-isomorphic)



Solution: Both non-isomorphic. On the left, one has a vertex with 4 neighbors but the other does not. On the right, one has two adjacent vertices with 2 neighbors each, while the other does not.

5: Find all non-isomorphic graphs on 3 vertices.

Solution:



6: Determine the number of different isomorphisms there are of the graph $K_{3,3}$ to itself.
These are called **automorphisms**. Try to define precisely what is automorphisms.

Solution: Automorphism is a bijective function $\varphi : V(G) \to V(G)$ such that$uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(G)$.
$6 \cdot 2! \cdot 3! = 72$.

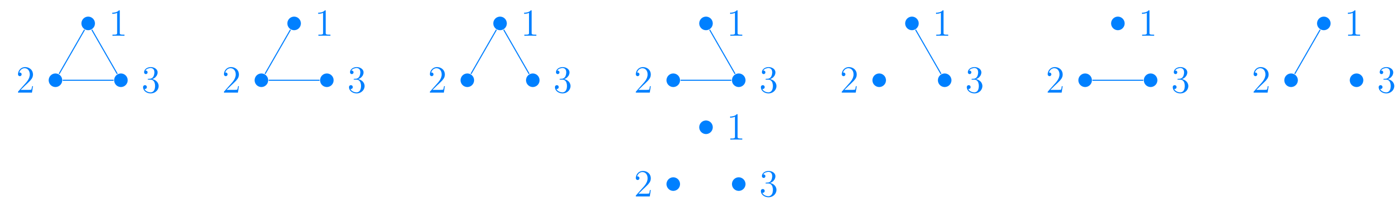
7: Find all non-isomorphic graphs on 4 vertices.

Solution: There are 11 of them.

Question: How many non-isomorphic graphs are there on n vertices?

8: Enumerate all graphs with $V = \{1, 2, 3\}$. (no isomorphism now)

Solution:



9: How many graphs are there with vertex set $V = \{1, 2, \dots, n\}$? (do not consider isomorphism now)

Solution: For every pair, we can decide if it is an edge or not. Since we have $\binom{n}{2}$ potential edges, we have $2^{\binom{n}{2}}$ possible graphs.

10: If G is a graph on n vertices, how many ways (at most) are there to change $V(G)$ to $\{1, 2, \dots, n\}$?

Solution: $n!$

11: Using the previous exercises, what is the lower bound on the number of non-isomorphic graphs on n vertices? Examine how fast it grows?

Solution:

$$\frac{2^{\binom{n}{2}}}{n!}$$

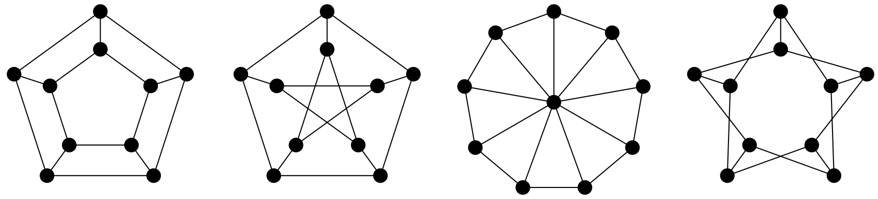
Now we use estimate $n! \le n^n$ and take the logarithm. It gives

$$1 \log_2 \frac{2^{\binom{n}{2}}}{n!} \ge \binom{n}{2} \log_2 2 - \log_2 n^n = \frac{1}{2}n^2 - \frac{1}{2}n - n \log n = \frac{n^2}{2}\left(1 - \frac{1}{n} - \frac{2\log_2 n}{n}\right)$$

So it grows almost like $2^{n^2/2}$ in the limit. Certainly larger than 2^n.

The number of non-isomorphic graphs of orders 0, 1, 2, 3, 4, 5,...:
1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 31426485969804308768, 64001015704527557894928, 245935864153532932683719776, 1787577725145611700547878190848, 24637809253125004524383007491432768
See `http://oeis.org/A000088`

12: Find which pairs of the following graphs are isomorphic.



G_1 G_2 G_3 G_4

Solution:

13: Give an example of three different (non-isomorphic) graphs of order 5 and size 5.

Solution:

14 *Open problem*: Find a polynomial time algorithm to determine if two graphs are isomorphic.